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Monodromy in two degrees of freedom integrable systems

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Geometric monodromy is an obstruction for the global existence of action variables. In this paper we develop a simple method for computing the monodromy operator for two degrees of freedom integrable systems near an isolated singularity of the energy momentum map. We show that under some nondegeneracy conditions, the monodromy operator can be determined from some local data of the energy momentum map at the singularity.

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1. Introduction

In the last decade or so, there has been interest in studying the global structure of the torus bundle defined by an integrable Hamiltonian system. The simplest problem is to determine whether or not this torus bundle is topologically trivial. It has been shown by Duistermaat [7] that there are two obstructions to the triviality of this bundle: the monodromy and the Chern class. For two degrees of freedom integrable systems near an isolated critical value of the energy momentum map, the only obstruction is the monodromy. It has been proved for some classical mechanical systems like the spherical pendulum [4,7], the Lagrange top [5] and the Kirchhoff case of the motion of a rigid body in an infinite ideal fluid [2], that the monodromy is nontrivial. However, the computation of the monodromy operator is a rather delicate matter and has always been case dependent. In this paper, we study the energy momentum map for two degrees of freedom systems near an isolated singularity, and show that under some nondegeneracy conditions, the monodromy of the torus bundle defined by the energy momentum map can be easily determined from local data at the singularity. Our main result is the following

Theorem 1.1. *Let $f=(f_1, f_2)$ be an integrable Hamiltonian system on (M^4, ω). Assume*

(1) C_f contains an isolated point $p=(0, 0)$. f is singular only at one point $x_0 \in f^{-1}(p)$;

(2) all the regular fibers $f^{-1}(r)$ for r near p are diffeomorphic to $\mathbb{T}^2$;

(3) there are canonical local coordinates (q_1, q_2, p_1, p_2) near x_0 in which f has the following normal form:

$$f=\begin{pmatrix} \frac{1}{2}a(p_1^2+p_2^2)-\frac{1}{2}b(q_1^2+q_2^2)+c(q_2 p_1-q_1 p_2)+\text{higher order terms} \\ q_2 p_1 - q_1 p_2 \end{pmatrix}, \quad (1)$$

where $a>0$, $b>0$ and c are constants.

Then for any fixed regular value r_0 near p and any simple positively orientated loop γ around p connecting r_0 to itself, the monodromy operator $h_\gamma: H_1(f^{-1}(r_0), \mathbb{Z}) \to H_1(f^{-1}(r_0), \mathbb{Z})$ can be represented, in some appropriate choice of basis for $H_1(f^{-1}(r_0), \mathbb{Z})$, as the matrix $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$.

This paper is organized as follows. In section2, we give some definitions and introduce some notations. In section 3, we prove theorem 1.1. We show, by using the stability theorem for equivariant maps, that f is locally left–right equivalent to its quadratic part in the space of maps invariant under the S^1 action generated by the Hamiltonian flow of its second component. Then we use the Ehresmann fibration theorem to reduce the problem of determining the monodromy operator to the problem of computing the variation of covanishing cycle for the simple map w^2-z^2. This variation can be computed easily by using the classical Picard–Lefschetz theorem [8]. In section 4, we present examples [which cover all the published systems exhibiting monodromy $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$]. We also give two examples where theorem 1.1 cannot be applied.

2. Some definitions

Throughout this article all the manifolds are assumed to be paracompact connected smooth manifolds, and all the maps are assumed to be C^∞ proper maps.

2.1. THE MONODROMY OPERATOR

Let M be a smooth manifold with boundary ∂M (∂M may be empty). Let N be a contractible smooth manifold. Let $f: M \to N$ be a smooth map so that its restriction to ∂M, $\partial f: \partial M \to N$ is a regular map (if $\partial M \neq \emptyset$, we assume also that ∂M consists of only finitely many connected components $\partial M_1, \ldots, \partial M_k$ and that $\partial f(\partial M_i) = f(M)$ for all $i=1, \ldots, k$.) Let R_f (C_f) denote the set of regular (critical) values of f and let r_0 be a fixed regular value. It follows from the Ehresmann fibration theorem [12] that ∂f defines a trivial fibration, that is, $(\partial f)^{-1}(N)$ is diffeomorphic to $(\partial f)^{-1}(r_0)\times N$. Fix a trivialization of this fibration $T: (\partial f)^{-1}(r_0)\times N\to(\partial f)^{-1}(N)$ and choose an Ehresmann connection $\mathscr{H}$ compatible with T (that is, when restricted to $(\partial f)^{-1}(r_0)$, parallel translation of $f^{-1}(r_0)$ with respect to $\mathscr{H}$ is the same as the trivialization T).

Let $\gamma\subset R_f$ be a piecewise smooth loop beginning and ending at r_0. By parallel translating $f^{-1}(r_0)$ along γ, one gets a smooth map $T_\gamma: f^{-1}(r_0)\to f^{-1}(r_0)$ satisfying $T_\gamma|_{(\partial f)^{-1}(r_0)}=\text{identity}$. The action h_γ of T_γ on the homology $H_*(f^{-1}(r_0), \mathbb{Z})$ is called the *monodromy operator of the loop* γ. It can be proved that h_γ is uniquely determined by the homotopy class of γ. The image of $h:\pi_1(R_f, r_0)\to\text{Aut}(H_*(f^{-1}(r_0), \mathbb{Z}))$ is called the *monodromy group* of the fibration with respect to r_0. Since T_γ is the identity map on $(\partial f)^{-1}(r_0)$, for any relative cycle δ the difference $T_\gamma\delta-\delta$ is an absolute cycle in $f^{-1}(r_0)$. The induced homomorphism

$$\text{var}_\gamma: H'_*((f^{-1}(r_0), (\partial f)^{-1}(r_0)), \mathbb{Z})\to H_*(f^{-1}(r_0), \mathbb{Z}) , \delta\mapsto T_\gamma\delta-\delta ,$$

is called the *variation operator of* γ.

2.2. THE PICARD–LEFSCHETZ THEOREM

Four our application, let us recall the simplest case of the Picard–Lefschetz theorem. (For a comprehensive treatment of this subject, see ref. [8].) Let

$$M^4=\{(w, z)\in\mathbb{C}^2: |w|+|z|<3\sqrt{\epsilon}, |w^2-z^2|<2\epsilon\} ,$$

$$N=\{y\in\mathbb{C}: |y|<2\epsilon\} .$$

Then $f=w^2-z^2: M\to N$ is a smooth proper map. All the conditions mentioned at the beginning of this section are satisfied. Moreover, $R_f=N\backslash 0$ and for all $r\in R_f$, $f^{-1}(r)$ is a cylinder. Let $r_0=\epsilon$ and $\tau(t)=\epsilon\exp(2\pi i t)$. Let Δ (∇) be the Picard–Lefschetz vanishing (covanishing) cycle on $f^{-1}(r_0)$. Then the variation operator var_τ is given by the Picard–Lefschetz formula: $\text{var}_\tau(\nabla)=-\Delta$. Moreover, h_τ is the identity map, i.e., $h_\tau\Delta=\Delta$.

3. Proof of theorem 1.1

Theorem 1.1 clearly follows from the following two lemmas.

Lemma 3.1. *Let* $f: M^4\to D_2\overset{\text{def}}{=}\{(x, y)\in\mathbb{R}^2, x^2+y^2<4\}$ *be a smooth proper map. Assume*

(1) f *has only one critical point* $x_0\in M^4$ *and* $f(x_0)=(0, 0)$;

(2) for all $r\in D_2\backslash\{(0, 0)\}$, $f^{-1}(r)$ *is diffeomorphic to* $\mathbb{T}^2$;

(3) there exist local coordinates (q_1, q_2, p_1, p_2) *near* x_0 *in which* f *is represented as*

$$\begin{pmatrix} p_1^2+p_2^2-q_1^2-q_2^2 \\ 2(q_2p_1-q_1p_2) \end{pmatrix}. \tag{2}$$

Let $\gamma(t)=(\cos 2\pi t, \sin 2\pi t)$. Then the monodromy operator h_γ is represented as the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ in some basis of $H_1(f^{-1}(1,0),\mathbb{Z})$.

Lemma 3.2. *Under the assumptions of theorem 1.1, there exist local coordinates (q_1,q_2,p_1,p_2) near x_0 and an origin preserving diffeomorphism $\psi:\mathbb{R}^2\to\mathbb{R}^2$ so that locally*

$$f=\psi\circ\begin{pmatrix} p_1^2+p_2^2-q_1^2-q_2^2 \\ 2(q_2p_1-q_1p_2) \end{pmatrix}. \tag{3}$$

We sketch the proofs of these two lemmas. The details can be found in ref. [14].

Proof of lemma 3.1. Let $z=q_1+ip_2$ and $w=p_1+iq_2$. Then in a small neighborhood U of x_0, f can be considered as the map $\tilde{f}=w^2-z^2: U\to V$, where $V\subset\mathbb{C}$ is a contractible neighborhood of $0\in\mathbb{C}$. Let $\tau\subset V$ be a small simple loop around 0 beginning at ϵ. Then by the Picard–Lefschetz theorem, we have $\mathrm{var}_\tau(\nabla)=-\Delta$ and $h_\tau\Delta=\Delta$. It is clear that Δ is one of the generators of $H_1(f^{-1}(\epsilon,0),\mathbb{Z})$. Extend ∇ to the other generator Δ_1 of $H_1(f^{-1}(\epsilon,0),\mathbb{Z})$. Write $\Delta_1=\nabla+\nabla_1$, where ∇_1 is a relative cycle in $(f|_{M^4\setminus U})^{-1}(\epsilon,0)$. Since $f|_{M^4\setminus U}$ is a regular map, by the Ehresmann fibration theorem it defines a trivial fibration. Thus, we can choose an Ehresmann connection for f compatible with the trivial bundle structure of $\partial f: (\partial f)^{-1}(D_2)\to D_2$, so that parallel translation of $f^{-1}(\epsilon)$ along τ preserves ∇_1. As a result, we have $h_\tau(\Delta_1)=\Delta_1-\Delta$. Thus in the basis $-\Delta$, Δ_1 of $H_1(f^{-1}(\epsilon),\mathbb{Z})$, the monodromy operator h_τ is represented by the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. Lemma 3.1 now clearly follows.

Before we prove lemma 3.2, let us first recall some facts from the theory of equivariant stability of maps. For a complete treatment of this theory, see refs. [3,10].

Let G be a compact Lie group acting linearly on $\mathbb{R}^n$. It is well known that the space of G-invariant polynomials is finitely generated. Let $\rho_1, \ldots, \rho_k$ be a set of Hilbert generators. By a theorem of Schwarz [11], any G-invariant smooth function on $\mathbb{R}^n$ can be written as a smooth function of $\rho_1, \ldots, \rho_k$.

Let $\mathscr{E}^G(\mathbb{R}^n,\mathbb{R}^p)$ denote the set of G-invariant germs from $(\mathbb{R}^n,0)$ to $(\mathbb{R}^p,0)$. Denote by $\mathrm{Diff}_0^G(\mathbb{R}^n)$ the set of G-equivariant local diffeomorphisms on $(\mathbb{R}^n,0)$ and by $\mathrm{Diff}_0(\mathbb{R}^p)$ the set of local diffeomorphisms on $(\mathbb{R}^p,0)$. There is a natural action of $\mathrm{Diff}_0^G(\mathbb{R}^n)\times\mathrm{Diff}_0(\mathbb{R}^p)$ on $\mathscr{E}^G(\mathbb{R}^n,\mathbb{R}^p)$ defined by $(\phi,\psi)\cdot f \stackrel{\mathrm{def}}{=} \psi\circ f\circ\phi^{-1}$.

A germ $f \in \mathscr{E}^G(\mathbb{R}^n, \mathbb{R}^p)$ is called *stable* if its orbit in $\mathscr{E}^G(\mathbb{R}^n, \mathbb{R}^p)$ is open. It is called *infinitesimally stable* if $\mathscr{E}^G(\mathbb{R}^n, \mathbb{R}^p) = \mathscr{J}(f) + (\mathscr{F}(f))^p$, where $\mathscr{J}(f)$ is the (equivariant) Jacobian module of f and $\mathscr{F}(f)$ is the module generated by the components of f. It is well known that the concept of stability and the concept of infinitesimal stability are equivalent.

Proof of lemma 3.2. By assumption 3 of theorem 1.1, we see that f is invariant under the S^1 action generated by the Hamiltonian flow of its second component. A set of Hilbert generators for this action is given by $X = q_1^2 + q_2^2$, $Y = p_1^2 + p_2^2$, $Z = 2(q_1 p_1 + q_2 p_2)$ and $S = 2(q_2 p_1 - q_1 p_2)$. There is one relation among these generators, namely $4XY = S^2 + Z^2$. To prove lemma 3.2, it is enough to show that f is stable in the space of S^1-invariant smooth maps. It is not hard to see that this assertion is equivalent to the following

Theorem 3.3. *The map* $g = \binom{Y-X}{S}$ *is stable in* $\mathscr{E}^{S^1}(\mathbb{R}^4, \mathbb{R}^2)$.

To prove this theorem, we need to check the condition $\mathscr{E}^{S^1}(\mathbb{R}^4, \mathbb{R}^2) = \mathscr{J}(g) + (\mathscr{F}(g))^2$ is satisfied. By some elementary calculations, one can show that $\mathscr{J}(g)$ is generated by

$$\epsilon_1 = \begin{pmatrix} Z \\ 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 \\ Z \end{pmatrix}, \quad \epsilon_3 = \begin{pmatrix} Y+X \\ 0 \end{pmatrix},$$

$$\epsilon_4 = \begin{pmatrix} 0 \\ Y+X \end{pmatrix}, \quad \epsilon_5 = \begin{pmatrix} S \\ 2X \end{pmatrix}, \quad \epsilon_6 = \begin{pmatrix} Y-X \\ S \end{pmatrix}.$$

By Schwarz' theorem, every S^1-invariant smooth function is of the form $f(Y+X, Y-X, Z, S)$. One can show easily, by using the relation $4XY = S^2 + Z^2$, that f can be written as $f = (Y+X)\tilde{f}_1 + Z\tilde{f}_2 + F(Y-X, S)$, where $\tilde{f}_i$ are S^1-invariant functions. From this, it is easy to see that $\mathscr{E}^{S^1}(\mathbb{R}^4, \mathbb{R}^2) = \mathscr{J}(g) + (\mathscr{F}(g))^2$ holds. □

4. Some examples

In this section, we give a few examples in which theorem 1.1 can be applied. In fact, all the known examples with monodromy $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are covered by theorem 1.1. We also give two examples where theorem 1.1 fails.

4.1. EXAMPLES IN WHICH THEOREM 1.1 APPLIES

4.1.1. The motion of a particle in the champagne bottle. The simplest mechanical system which has monodromy is the motion of a point mass in an S^1-symmetric double well [1]. The energy momentum map for this system is given by

$f=(H,L)$, where

$$H=\tfrac{1}{2}(p_x^2+p_y^2)-(x^2+y^2)+(x^2+y^2)^2\,,\qquad L=yp_x-xp_y\,.$$

The isolated critical point of f is the unstable equilibrium of the particle which corresponds to the isolated critical value $(0,0)$. It is straightforward to check that all the conditions required by theorem 1.1 are met. Therefore, we conclude that the monodromy group of the torus bundle $E=(H,L):\mathbb{R}^4\to\mathbb{R}^2$ is generated by the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$.

4.1.2. The spherical pendulum. The spherical pendulum was the first mechanical system discovered to have monodromy [4,7]. The phase space for this system is

$$TS^2=\{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^6:\boldsymbol{x}\cdot\boldsymbol{x}=1,\,\boldsymbol{x}\cdot\boldsymbol{y}=0\}\,.$$

The energy momentum map is given by $f=(H,L):TS^2\to\mathbb{R}^2$, where

$$H=\tfrac{1}{2}(y_1^2+y_2^2+y_3^2)+x_3\,,\qquad L=x_2y_1-x_1y_2\,.$$

As in the last example, the isolated critical point is the unstable equilibrium which corresponds to the isolated critical value $(1,0)$. It is not hard to check that all the conditions of theorem 1.1 are satisfied in the following local canonical coordinates near the unstable equilibrium:

$$q_1=x_1\,,\quad q_2=x_2\,,\quad p_1=\frac{x_1y_3-x_3y_1}{x_3}\,,\quad p_2=\frac{x_2y_3-x_3y_2}{x_3}\,.$$

Hence we see that the monodromy operator for any simple positive loop around $(1,0)$ is given by the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$.

4.1.3. The Lagrange top. The Lagrange top is an axially symmetric rigid body fixed at a stationary point in a uniform force field. The motion of the body can be described (with some simplifications) in the following way [9].

Define a bracket on $\mathbb{R}^6$ by putting

$$\{y_i,y_j\}=\epsilon_{ijk}y_k\,,\qquad\{y_i,x_j\}=\epsilon_{ijk}x_k\,,\qquad\{x_i,x_j\}=0\,,\quad i,j,k=1,2,3\,,$$

and extend $\{\cdot,\cdot\}$ to $C^\infty(\mathbb{R}^6)$ by the Leibniz rule. It is easy to check that $(C^\infty(\mathbb{R}^6),\cdot,\{\ \})$ is a Poisson algebra. Notice that there are two Casimirs for this bracket, namely, $F_1=\boldsymbol{x}\cdot\boldsymbol{x}$ and $F_2=\boldsymbol{x}\cdot\boldsymbol{y}$. The Lagrange top can be thought of as a two degrees of freedom system on the symplectic leaf $\mathcal{O}_b=\{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^6:\boldsymbol{x}\cdot\boldsymbol{x}=1,\,\boldsymbol{x}\cdot\boldsymbol{y}=b\}$. Here we think of b as a parameter. Physically, b is the total momentum of the body.

The energy momentum map for the Lagrange top is $f=(H,L):\mathcal{O}_b\to\mathbb{R}^2$, where $H=\frac{1}{2}(y_1^2+y_2^2+Ay_3^2)+x_3$ and $L=y_3$, with $A>\frac{1}{2}$ the ratio of the unequal moments of inertia.

It can be shown [5] that for $|b|<2$, the energy momentum map f has an isolated critical value $(h_0, l_0) = (\frac{1}{2}Ab^2+1, b)$ and that the unstable equilibrium is the only singular point of f on $f^{-1}(h_0, l_0)$. Moreover, all the regular level surfaces are $\mathbb{T}^2$. In order to apply theorem 1.1, we need to find local canonical coordinates so that $f=(H, L)$ is of the required normal form. These local canonical coordinates are given by

$$q_1 = x_1 , \qquad q_2 = x_2 , \qquad p_1 = -\frac{1}{x_3}\left(y_2 - \frac{bx_2}{1+x_3}\right), \qquad p_2 = \frac{1}{x_3}\left(y_1 - \frac{bx_1}{1+x_3}\right).$$

Thus, we see that for $|b|<2$, the torus bundle $f=(H, L): \mathcal{O}_b \to \mathbb{R}^2$ has monodromy and the monodromy operator can be represented by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

4.2. EXAMPLES FOR WHICH THEOREM 1.1 FAILS

Theorem 1.1 requires, besides the S^1 symmetry, the following nondegeneracy conditions:

(1) the singular values of the energy momentum map f are distinct;

(2) the quadratic part of the local normal form of f is stable.

When either of these two conditions is not met, theorem 1.1 fails. We now give two examples for which one of these conditions is violated.

4.2.1. The motion of a particle in a degenerate champagne bottle. Consider the two Poisson commuting functions on $\mathbb{R}^4$

$$f_1 = \tfrac{1}{2}(p_1^2 + p_2^2) - (q_1^2 + q_2^2)^2 + (q_1^2 + q_2^2)^3 , \qquad f_2 = q_2 p_1 - q_1 p_2 .$$

This system describes the motion of a point mass on the plane under the influence of the S^1-symmetric potential $V = -(q_1^2+q_2^2)^2 + (q_1^2+q_2^2)^3$. As in the champagne bottle example, one can check easily that the first two conditions of theorem 1.1 are satisfied. However, since the quadratic part of f is unstable (in fact, one can show that f is (2, 1) determined in $\mathscr{E}^{S^1}(X, Y, Z, S)$ [6,p.63], theorem 1.1 cannot be applied. Nevertheless, one can show that this system has monodromy and the monodromy operator can be represented as the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

4.2.2. The square potential spherical pendulum. The square potential spherical pendulum is the motion of a particle on the unit sphere S^2 subject to a vertical force field of magnitude $-x_3$. As in the case of the spherical pendulum, the motion can be described as a Hamiltonian system on TS^2. The Hamiltonian function is given by $H = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + \frac{1}{2}x_3^2$. The other constant of motion is the angular momentum $L = x_2 y_1 - x_1 y_2$.

For the energy momentum map $f=(H, L)$, one can check easily that $(h, l) = (\frac{1}{2}, 0)$ is an isolated singular value of f and that all the regular level surfaces of f are diffeomorphic to $\mathbb{T}^2$. However, since f is singular at two distinct points on $f^{-1}(\frac{1}{2},$

0), that is, the singular values of the energy momentum map are not distinct, we cannot apply theorem 1.1.

By computing explicitly the action variables for this system, one can show that the torus bundle defined by f has monodromy and that the monodromy operator can be represented as the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (see refs. [2,13]).

Remark. As a rule, the degeneracy in these two examples can be removed by deformations. In the first example, one can show that a versal deformation of the energy momentum map is given by [6,p.64]

$$f_\nu = \begin{pmatrix} \frac{1}{2}(p_1^2+p_2^2)+\nu(q_1^2+q_2^2)-(q_1^2+q_2^2)^2+(q_1^2+q_3^2)^3 \\ q_2 p_1 - q_1 p_2 \end{pmatrix}.$$

For small negative ν, theorem 1.1 is applicable. We remark that for small positive ν, the torus bundle defined by f_ν also has monodromy. For the second example, a small linear perturbation of the potential will split the singular value into two isolated ones. For each of them, theorem 1.1 can be applied.

To make this deformation argument rigorous, one has to show that the monodromy does not depend on the deformation. This will be discussed in a later paper.

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